15. (a) If f and g are bounded functions in $\alpha[a,b]$, then show that $f+g$ is in $L[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Or

(b) Let $f \in L[a,b]$. Show that given $E>0$ there exists $f>0$ such that $\left|\int_E f\right| < \in$ whenever E is a measurable subset of $[a,b]$ with $mE<f$.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. State and prove the Weierstrass Approximation Theorem.

17. Explain the Chain Rule.

18. State and prove the Inverse function theorem.

19. If E_1 and E_2 are subsets of $[a.b]$, prove that $\overline{m}E_1 + \overline{m}E_2 \geq \overline{m}(E_1 \cup E_2) + \overline{m}(E_1 \cap E_2)$ and $\underline{m}E_1 + \underline{m}E_2 \leq \underline{m}(E_1 \cup E_2) + \underline{m}(E_1 \cap E_2)$.

20. Derive Lebesgue dominated convergence theorem.

NOVEMBER/DECEMBER 2024

DMA22 — REAL ANALYSIS – II

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Time : Three hours Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Define Fourier Series.

2. State Riemann's localization theorem.

3. Define Directional Derivative.

4. What is the sufficient condition for differentiability?

5. Define Jacobian determinant.

6. What is meant by Saddle Point?

7. Define the length of an open set.

8. What are the properties of measurable functions?

9. Define Lebesque integral for unbounded functions.

10. What is Fatou's Lemma?

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be orthonormal on I and assume that $f \in L^2(I)$. Define $\{S_n\}$ and $\{t_n\}$ on I as $S_n(x) = \sum_{k=0}^{n} C_K \phi_K(x)$, $t_n(x) = \sum_{K=0}^{n} b_K \phi_K(x)$, $C_K = (f, \phi_K)$ for $K = 0,1,2,\ldots$ and $b_0, b_1, b_2, \ldots$ are arbitary complex numbers. Prove that for each n, $\|f - S_n\| \leq \|f - t_n\|$ iff $b_K = C_K$ for $K = 0, 1, \ldots, n$.

Or

(b) If g is of bounded variation on $[0, f]$, prove that $\lim_{\alpha \to +\infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin \alpha t}{t} dt = g(0+)$.

12. (a) If f is differentiate at c, prove that f is continuous at c.

Or

(b) State and prove Mean–Value Theorem.

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13. (a) Assume that $f = (f_1, \ldots, f_n)$ has continuous partial derivatives $D_j f_i$ on an open set S in R^n and that the Jacobian determinant $J_f(a) \neq 0$ for some point a in S. Prove that there is an n–ball $B(a)$ on which f is 1–1.

Or

(b) Let f be a real–valued function with continuous second–order partial derivatives at a stationary point 'a' in R^2. Let $A = D_{1,1} f(a)$, $B = D_{1,2} f(a)$, $C = D_{2,2} f(a)$, $\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$. Prove that

(i) If $\Delta > 0$ and $A > 0$, f has a relative minimum at a

(ii) If $\Delta > 0$ and $A < 0$, f has a relative maximum at a

(iii) If $\Delta < 0$, f has a saddle point at a.

14. (a) If $G_1, G_2, \ldots$ are open subsets of $[a,b]$, prove that $\left| \bigcup_{n=1}^{\infty} G_n \right| \leq \sum_{n=1}^{\infty} |G_n|$.

Or

(b) If f is a measurable function $[a,b]$ and if $C \in R$, prove that the functions $f + c$ and cf are measurable.